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A NOTE ON SYSTEM TRUNCATION

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A NOTE ON SYSTEM TRUNCATION

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FOREWORD

This document is one of sixteen sections that comprise the final report prepared by the Minneapolis-Honeywell Regulator Company for the National Aeronautics and Space Administration under contract NASw-563. The report is issued in the following sixteen sections to facilitate updating as progress warrants:

1541-TR 1	Summary
1541-TR 2	Control of Plants Whose Representation Contains Derivatives of the Control Variable
1541-TR 3	Modes of Finite Response Time Control
1541-TR 4	A Sufficient Condition in Optimal Control
1541-TR 5	Time Optimal Control of Linear Recurrence Systems
1541-TR 6	Time-Optimal Bounded Phase Coordinate Control of Linear Recurrence Systems
1541-TR 7	Penalty Functions and Bounded Phase Coordinate Control
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1541-TR 10	A Concise Formulation of a Bounded Phase Coordinate Control Problem as a Problem in the Calculus of Variations
1541-TR 11	A Note on System Truncation
1541-TR 12	State Determination for a Flexible Vehicle Without a Mode Shape Requirement
1541-TR 13	An Application of the Quadratic Penalty Function Criterion to the Determination of a Linear Control for a Flexible Vehicle
1541-TR 14	Minimum Disturbance Effects Control of Linear Systems with Linear Controllers
1541-TR 15	An Alternate Derivation and Interpretation of the Drift-Minimum Principle
1541-TR 16	A Minimax Control for a Plant Subjected to a Known Load Disturbance

Section 1 (1541-TR 1) provides the motivation for the study efforts and objectively discusses the significance of the results obtained. The results of inconclusive and/or unsuccessful investigations are presented. Linear programming is reviewed in detail adequate for sections 6, 8, and 16.

It is shown in section 2 that the purely formal procedure for synthesizing an optimum bang-bang controller for a plant whose representation contains derivatives of the control variable yields a correct result.

In section 3 it is shown that the problem of controlling m components ($1 < m \leq n$), of the state vector for an n-th order linear constant coefficient plant, to zero in finite time can be reformulated as a problem of controlling a single component.

Section 4 shows Pontriagin's Maximum Principle is often a sufficient condition for optimal control of linear plants.

Section 5 develops an algorithm for computing the time optimal control functions for plants represented by linear recurrence equations. Steering may be to convex target sets defined by quadratic forms.

In section 6 it is shown that linear inequality phase constraints can be transformed into similar constraints on the control variables. Methods for finding controls are discussed.

Existence of and approximations to optimal bounded phase coordinate controls by use of penalty functions are discussed in section 7.

In section 8 a maximum principle is proven for time-optimal control with bounded phase constraints. An existence theorem is proven. The problem solution is reduced to linear programming.

A backing-out-of-the-origin procedure for obtaining trajectories for time-optimal control with amplitude and rate limited control variables is presented in section 9.

Section 10 presents a reformulation of a time-optimal bounded phase coordinate problem into a standard calculus of variations problem.

A mathematical method for assessing the approximation of a system by a lower order representation is presented in section 11.

Section 12 presents a method for determination of the state of a flexible vehicle that does not require mode shape information.

The quadratic penalty function criterion is applied in section 13 to develop a linear control law for a flexible rocket booster.

In section 14 a method for feedback control synthesis for minimum load disturbance effects is derived. Examples are presented.

Section 15 shows that a linear fixed gain controller for a linear constant coefficient plant may yield a certain type of invariance to disturbances. Conditions for obtaining such invariance are derived using the concept of complete controllability. The drift minimum condition is obtained as a specific example.

In section 16 linear programming is used to determine a control function that minimizes the effects of a known load disturbance.

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A NOTE ON SYSTEM TRUNCATION*

By E. R. Rang‡

ABSTRACT

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The problem of control design of high order systems based on the characteristics of the lower order essential elements of the system is considered by approximation of asymptotic, sometimes called singular perturbation, type. A preliminary general development is given and detail calculations for a simple problem are written out. These results indicate that the technique leads to the same design found by previous investigators and the only advantage appears to be an estimate of errors. Author

INTRODUCTION

A large number of problems devolve to an estimation of solutions of differential equations of high order from solutions of equations of much lower order obtained by "truncating" the original system. A general analytical estimating procedure for the case in which the system of equations has constant coefficients is outlined, and a particular example is investigated.

GENERAL DEVELOPMENT

Suppose that the system can be written in the form

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* Prepared under contract NASw-563 for the NASA.

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$$\dot{x} = A(\epsilon)x + B(\epsilon)y + a(\epsilon)\, u(t)$$
$$\epsilon\dot{y} = C(\epsilon)x + D(\epsilon)y + b(\epsilon)\, u(t) \tag{1}$$

in which the vector x has m components, y has n components, and u(t) is a single forcing function. The constant matrices $A(\epsilon)$, $B(\epsilon)$, $C(\epsilon)$, $D(\epsilon)$ and vectors $a(\epsilon)$, $b(\epsilon)$ are of compatible dimensions, of course, and differentiable in ϵ; ϵ is assumed to be a positive scalar parameter which will be regarded as small. In applications, it will be the reciprocal of some gain parameter.

If $\epsilon \longrightarrow 0$, equations (1) reduce to a system of order m,

$$\varphi = A(0)\, \varphi + B(0)\psi + a(0)\, u(t)$$
$$0 = C(0)\, \varphi + D(0)\psi + b(0)\, u(t), \tag{2}$$

which will be called the reduced system. Now assume $\det D(0) \neq 0$ so that the second equation of (2) may be solved to give

$$\psi = -\, D^{-1}(0)[C(0)\, \varphi + b(0)\, u(t)], \tag{3}$$

$$\dot{\varphi} = [A(0) - B(0)D^{-1}(0)C(0)]\, \varphi + [a(0) - B(0)D^{-1}(0)b(0)]u(t) \tag{4}$$

or

$$\dot{\varphi} = F(0)\, \varphi + f(0)u(t),\ \varphi(0) = \varphi_o, \tag{5}$$

where

$$F(0) = A(0) - B(0)D^{-1}(0)C(0) \text{ and } f(0) = a(0) - B(0)D^{-1}(0)b(0).$$

Suppose now, that a function u(t) which takes φ from an initial point φ_o to the origin time-optimally or according to some other criterion has been found. The problem is to estimate the behavior of the solutions x, y of equation (1) which initiate from x_o, y_o. Hence, a formula in ϵ for the errors $\xi = x - \varphi, \eta = y - \psi$, starting with initial conditions

$$\xi_o = x_o - \varphi_o$$
$$\eta_o = y_o - \psi_o = y_o + D^{-1}(0)[C(0)\varphi_o + b(0)\ u(0)], \qquad (6)$$

is sought.

ERROR EQUATIONS

The differential equations for the errors are

$$\begin{aligned}
\dot{\xi} &= A(\epsilon)\xi + B(\epsilon)\eta + \left\{A(\epsilon)-A(0)-[B(\epsilon)-B(0)]D^{-1}(0)C(0)\right\}\varphi \\
&\quad + \left\{a(\epsilon) - a(0) - [B(\epsilon)-B(0)]D^{-1}(0)b(0)\right\}u(t) \\
\epsilon\dot{\eta} &= C(\epsilon)\xi + D(\epsilon)\eta + \left\{C(\epsilon) - D(\epsilon)D^{-1}(0)C(0)\right. \\
&\quad \left. + \epsilon D^{-1}(0)C(0)\ F(0)\right\}\varphi + \left\{b(\epsilon) - D(\epsilon)D^{-1}(0)b(0)\right. \\
&\quad \left. + \epsilon D^{-1}(0)C(0)\ f(0)\right\}u(t) + \epsilon D^{-1}(0)b(0)\dot{u}(t).
\end{aligned} \qquad (7)$$

But with relay control, $\dot{u}(t)$ does not exist everywhere. Hence, a transformation

$$\eta = \zeta - D^{-1}(\epsilon)C(\epsilon)\xi + D^{-1}(0)b(0)\ u(t) \qquad (8)$$

is made (the second term is added to introduce a factor of ϵ in the ξ-term), and the error equations become

$$\begin{aligned}
\dot{\xi} &= F(\epsilon)\xi + B(\epsilon)\zeta + G(\epsilon)\varphi + g(\epsilon)\ u(t) \\
\epsilon\dot{\zeta} &= \epsilon D^{-1}(\epsilon)C(\epsilon)F(\epsilon)\xi + [D(\epsilon) + \epsilon D^{-1}(\epsilon)C(\epsilon)B(\epsilon)]\ \zeta \\
&\quad + H(\epsilon)\ \varphi + h(\epsilon)\ u(t),
\end{aligned} \qquad (9)$$

where the abbreviations

$$\begin{aligned}
F(\epsilon) &= A(\epsilon) - B(\epsilon)D^{-1}(\epsilon)C(\epsilon) \\
G(\epsilon) &= A(\epsilon) - A(0) - [B(\epsilon)-B(0)]D^{-1}(0)C(0) \\
g(\epsilon) &= a(\epsilon) - a(0) + B(0)\ D^{-1}(0)b(0) \\
H(\epsilon) &= C(\epsilon) - D(\epsilon)D^{-1}(0)C(0) + \epsilon D^{-1}(0)C(0)F(0)+ \\
&\quad + \epsilon D^{-1}(\epsilon)C(\epsilon)G(\epsilon) \\
h(\epsilon) &= b(\epsilon) + \epsilon D^{-1}(0)C(0)f(0) + \epsilon D^{-1}(\epsilon)C(\epsilon)g(\epsilon).
\end{aligned} \qquad (10)$$

have been introduced. The solution which is desired has the initial conditions

$$\begin{aligned} \xi_o &= x_o - \varphi_o \\ \zeta_o &= \eta_o + D^{-1}(\epsilon)C(\epsilon)\xi_o - D^{-1}(0)b(0)u(0) \\ &= y_o + D^{-1}(\epsilon)C(\epsilon)(x_o-\varphi_o) + D^{-1}(0)C(0)\varphi_o. \end{aligned} \tag{11}$$

THE CONSTRUCTION

Solutions of equation (9) of the form

$$\begin{aligned} \xi(t,\epsilon) &= p(t,\epsilon) + \epsilon P(t,\epsilon) \\ \zeta(t,\epsilon) &= q(t,\epsilon) + \epsilon Q(t,\epsilon) \end{aligned} \tag{12}$$

are sought. After substituting these in (9), powers of ϵ are identified so that the functions p, q, P, Q are required to satisfy

$$\begin{aligned} \dot{p} &= F(0)\, p + B(0)q + B(0)D^{-1}(0)b(0)u(t) \\ \epsilon\dot{q} &= D(\epsilon)q + b(0)\, u(t) \end{aligned} \tag{13}$$

$$\begin{aligned} \dot{P} &= F(0)P + B(0)Q + \frac{F(\epsilon)-F(0)}{\epsilon}(p+\epsilon P) + \frac{B(\epsilon)-B(0)}{\epsilon}(q+\epsilon Q) \\ &\quad + G(\epsilon)\varphi + \frac{a(\epsilon)-a(0)}{\epsilon}u(t) \\ \epsilon\dot{Q} &= D(\epsilon)Q + D^{-1}(\epsilon)C(\epsilon)F(\epsilon)(p+\epsilon P) + \epsilon D^{-1}(\epsilon)C(\epsilon)B(\epsilon)Q \\ &\quad + H(\epsilon)\varphi + \frac{h(\epsilon)-b(0)}{\epsilon}u(t) \end{aligned} \tag{14}$$

with the initial conditions

$$\begin{aligned} p_o &= x_o - \varphi_o \\ q_o &= y_o + D^{-1}(\epsilon)C(\epsilon)x_o \\ P_o &= 0 \\ Q_o &= \frac{1}{\epsilon}[D^{-1}(0)C(0) - D^{-1}(\epsilon)C(\epsilon)]y_o. \end{aligned} \tag{15}$$

Then equation (5) may be solved for φ, the second equation of (13) solved for q, the first for p, and finally the higher order errors from (14) may be estimated.

Let $\Phi(t)$ be a funamental matrix of the homogeneous part of (5), viz.,

$$\dot{\Phi} = F(0)\ \Phi, \quad \Phi(0) = I, \tag{16}$$

and $\Psi(t,\epsilon)$ be the corresponding result for the second equation of (13):

$$\dot{\Psi} = \frac{1}{\epsilon} D(\epsilon)\Phi, \quad \Psi(0,\epsilon) = I. \tag{17}$$

Then

$$\begin{aligned}
\varphi(t) &= \Phi(t)\varphi_o + \int_o^t \Phi(t-\tau)f(0)\ u(\tau)d\tau \\
q(t) &= \Psi(t,\epsilon)q_o + \frac{1}{\epsilon}\int_o^t \Psi(t-\tau,\epsilon)b(0)\ u(\tau)d\tau \\
p(t) &= \Phi(t)p_o + \int_o^t \Phi(t-\tau)\ [B(0)q(\tau) + B(0)D^{-1}(0)b(0)u(\tau)]d\tau,
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
P(t) &= \Phi(t)P_o + \int_o^t \Phi(t-\tau)\ J(\tau)d\tau \\
Q(t) &= \Psi(t,\epsilon)Q_o + \frac{1}{\epsilon}\int_o^t \Psi(t-\tau,\epsilon)\ K(\tau)d\tau,
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
J(t) &= B(0)Q(t)+ \frac{F(\epsilon)-F(0)}{\epsilon}\ [p(t)+\epsilon P(t)] + \frac{B(\epsilon)-B(0)}{\epsilon}[q(t)+\epsilon Q(t)] \\
K(t) &= D^{-1}(\epsilon)C(\epsilon)F(\epsilon)[p(t)+\epsilon P(t)] +\epsilon D^{-1}(\epsilon)C(\epsilon)B(\epsilon)Q(t) \\
&\quad + H(\epsilon)\quad (t) + \frac{h(\epsilon)-h(0)}{\epsilon}\ u(t).
\end{aligned} \tag{20}$$

CONVERGENCE

Sufficient hypotheses must be added so that

$$\left.\begin{array}{l} \xi(t,\epsilon) \xrightarrow[\epsilon \to 0^+]{} 0 \\ \eta(t,\epsilon) \xrightarrow[\epsilon \to 0^+]{} 0 \end{array}\right\} \quad (t > 0). \tag{21}$$

These imply $\zeta(t,\epsilon) \xrightarrow[\epsilon \to 0^+]{} -D^{-1}(0)b(0)u(t)$, $(t > 0)$. Certainly the convergence will not be uniform at $t = 0$, if the initial conditions ξ_o, η_o are not zero. There will be sort of a boundary layer here. Also trouble at the switching points of $u(t)$ is expected since $\zeta(t,\epsilon)$ will be continuous in t.

Now the basic supposition is made:

Hypothesis A: The characteristic roots of the matrix $D(\epsilon)$ have negative real parts. More precisely, if $d_i(\epsilon) = \alpha_i(\epsilon) + \sqrt{-1}\,\beta_i(\epsilon)$ are those characteristic roots, there is a constant α such that

$$\alpha_i(\epsilon) \leq -\alpha < 0, \; i = 1,2,\ldots,n.$$

Thus, the part of the system which is thrown away in the truncation is required to be stable and remain stable as $\epsilon \to 0^+$. With this assumption and the norm M of a matrix M as the sum of the absolute values of all its elements, a constant R can be found such that

$$\|\Psi(t,\epsilon)\| = \| e^{\frac{1}{\epsilon}D(\epsilon)t} \| < Re^{-\frac{\alpha}{\epsilon}t}; \; t \geq 0, \; \epsilon \geq 0. \tag{22}$$

This will be a basic tool for making estimates.

The first victim will be the integral in the second equation of (18). Assume that $u(t)$ has at most a finite number of jump discontinuities. If t is not one of these points, it is easy to show that

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^t \Psi(t-\tau,\epsilon)b(0)u(\tau)d\tau = -D^{-1}(0)b(0)u(t). \tag{23}$$

In any event the limit is $-D^{-1}(0)b(0)u(t-0)$.

Calculate as follows:

$$\frac{1}{\epsilon}\int_0^t \Psi(t-\tau,\epsilon)b(0)u(\tau)d\tau = D^{-1}(\epsilon)[\Psi(t,\epsilon)-I]b(0)u(t-\nu) + \frac{1}{\epsilon}\int_0^t \Psi(t-\tau,\epsilon)b(0)[u(\tau)-u(t-\nu)]d\tau,$$

where ν is chosen so that $u(\tau)$ is continuous on $[t-\nu,t)$. Let $0 < \nu < t$, and write the last term as

$$\left\|\frac{1}{\epsilon}\int_0^t\right\| \leq \left\|\frac{1}{\epsilon}\int_0^{t-\nu}\right\| + \left\|\frac{1}{\epsilon}\int_{t-\nu}^t\right\|.$$

For the first integral, with $|u| \leq U$,

$$\left\|\frac{1}{\epsilon}\int_0^{t-\nu}\right\| \leq \frac{1}{\epsilon} R\|b(0)\|\ 2U\int_0^{t-\nu} e^{-\frac{\alpha}{\epsilon}(t-\tau)}d\tau \leq \frac{2R\|b(0)\|U}{\alpha}\ e^{-\frac{\alpha}{\epsilon}\nu}.$$

For the second quantity, since $u(\tau) - u(t-\nu)$ is continuous in $[t-\nu,t)$, apply the mean value theorem to

$$\left\|\frac{1}{\epsilon}\int_{t-\nu}^t\right\| \leq \frac{1}{\epsilon}R\|b(0)\| \int_{t-\nu}^t e^{-\frac{\alpha}{\epsilon}(t-\tau)}|u(\tau)-u(t-\nu)|d\tau.$$

Hence, it is asserted that there is a t_1, $t-\nu < t_1 < t$ such that

$$\frac{1}{\epsilon}\int_{t-\nu}^t e^{-\frac{\alpha}{\epsilon}(t-\tau)}|u(\tau)-u(t-\nu)|d\tau = |u(t_1)-u(t-\nu)|\int_{t-\nu}^t \frac{1}{\epsilon}\,e^{-\frac{\alpha}{\epsilon}(t-\tau)}d\tau$$

$$\leq \frac{1}{\alpha}\ |u(t_1) - u(t-\nu)|.$$

Finally, with $\nu = \sqrt{\epsilon}$, let $\epsilon \longrightarrow 0^+$ and conclude that

$$\left\|\frac{1}{\epsilon}\int_0^t \Psi(t-\tau,\epsilon)b(0)\ [u(\tau)-u(t-\nu)]\ d\tau\right\| \longrightarrow 0.$$

The rest follows since $\|\Psi(t,\epsilon)\| \xrightarrow[\epsilon\to 0^+]{} 0$, $(t > 0)$.

Now turning to the integral in the third of equations 18, it can be shown that

$$\lim_{\epsilon \to 0^+} p(t) = \Phi(t)\, p_o, \ (t > 0). \tag{24}$$

Equations (19) may be analyzed with similar considerations since over a finite time interval P, Q are bounded. A conjecture is

$$\lim_{\epsilon \to 0^+} Q(t) = - D^{-1}(0) \lim_{\epsilon \to 0^+} K(t-0). \tag{25}$$

In any case, with sufficient patience, explicit formulas for the bounds on P, Q may be calculated. It is probably wiser to calculate these bounds for each example rather than to work out a general formula.

REMARKS

This calculation appears to be a complicated way of doing straightforward analysis since the equations considered have constant coefficients and explicit solutions may be immediately written. Nevertheless, it does have some merit in that the detailed structure of the coefficient matrix does not need to be considered to arrive at qualitative statements about the solution behavior. This procedure may be generalized to time-varying and nonlinear equations, and perhaps will be of greater utility in these cases. Nevertheless, it has been shown that a particular truncation process can be analytically formulated and qualitative estimates of the system behavior can be obtained.

The preceding calculations are summarized by the following formulas: the solutions of equations (1) with initial conditions x_o, y_o can be estimated in terms of the solutions of the reduced equations (2) with initial conditions φ_o, ψ_o by

$$
\begin{aligned}
x(t) &= \varphi(t) + p(t,\epsilon) + \epsilon P(t,\epsilon) \\
y(t) &= \psi(t) + D^{-1}(0)b(0)u(t) - D^{-1}(\epsilon)C(\epsilon)[p(t,\epsilon)+\epsilon P(t,\epsilon)] \\
&\qquad + q(t,\epsilon) + \epsilon Q(t,\epsilon),
\end{aligned}
\tag{26}
$$

and in the event that the roots of $D(\epsilon)$ are stable and $x_o = \varphi_o$, then

$$
\begin{aligned}
&\lim_{\epsilon \to 0^+} x(t) = \varphi(t) \qquad [t > 0,\ t = \text{points of discontinuity of } u(t)]. \\
&\lim_{\epsilon \to 0^+} y(t) = \psi(t),
\end{aligned}
\tag{27}
$$

AN EXAMPLE OF ASYMPTOTIC APPROXIMATION

Consider a simple third order plant which has a transfer function given by the expression

$$
\frac{k}{s(\tau s+1)(\mu s+1)}
\tag{28}
$$

where $\mu \ll \tau$. An attempt is made to regulate this plant time-optimally but with a second order controller since it is reasonable that the factor $(\mu s+1)$ will have little effect. The design of the proper controller is answered by J. A. Lovingood in reference 1. However, that design requires that the entire state of the third order system be measured or else an approximation must be made the consequences of which are not easily determined.

A search is made for an approximate controller using asymptotic analysis more or less along the lines of the previous general discussion. The results are compared with the mechanization given by Lovingood's procedure.

DEVELOPMENT

The differential equation associated with the transfer function

$$x(t) = kt + A + Be^{-t/\tau} + Ce^{-t/\mu} \tag{29}$$

when the plant is assumed to be driven by a plus one forcing function. The particular solution originating from the point $(x_o, \dot{x}_o, \ddot{x}_o)$ at time $t=0$ is

$$x(t) = kt + [x_o - (\tau+\mu)(k-\dot{x}_o) + \tau\mu\, \ddot{x}_o] + \frac{\tau^2}{\tau-\mu}[k-\dot{x}_o-\mu\, \ddot{x}_o]e^{-t/\tau} - \frac{\mu^2}{\tau-\mu}[k-\dot{x}_o-\tau\, \ddot{x}_o]e^{-t/\mu}, \tag{30}$$

and its derivative is found to be

$$\dot{x}(t) = k - \frac{\tau}{\tau-\mu}[k-\dot{x}_o - \mu\, \ddot{x}_o]e^{-t/\tau} + \frac{\mu}{\tau-\mu}[k-\dot{x}_o-\tau\ddot{x}_o]e^{-t/\mu} \tag{31}$$

For $t > \gamma > 0$, where γ is a small constant, the exponential $e^{-t/\mu}$ will be very small for small values of $\mu > 0$. In fact, there is a suitable constant A_n such that

$$e^{-t/\mu} < A_n\mu^n, \quad (t \geq \gamma) \tag{32}$$

for all positive integers n. We say that $e^{-t/\mu}$ is asymptotic to zero and write

$$e^{-t/\mu} \approx 0, \ (t \geq \gamma, \ (\mu \longrightarrow 0). \tag{33}$$

Hence (30) and (31) may be approximated asymptotically by the relations

$$x(t) \approx kt + [x_o - (\tau+\mu)(k-\dot{x}_o) + \tau\mu\ddot{x}_o] + \frac{\tau^2}{\tau-\mu}[k-\dot{x}_o-\mu\ddot{x}_o]e^{-t/\tau} \tag{34}$$

$$\dot{x}(t) \approx k - \frac{\tau}{\tau-\mu}[k-\dot{x}_o-\mu\ddot{x}_o]e^{-t/\tau}$$

and by combining the two expressions we get

$$x(t) \approx -\tau k \log\left[\frac{\tau-\mu}{\tau} \cdot \frac{k-\dot{x}(t)}{k-\dot{x}_o-\mu\ddot{x}_o}\right] + [x_o-(\tau+\mu)(k-\dot{x}_o)+\tau\mu\ddot{x}_o] + \tau[k-\dot{x}(t)]. \tag{35}$$

Repeating the definition of asymptotics, all that is meant by relation (35), which is of the form

$$f(t) \approx \varphi(t), \tag{36}$$

is that there is a constant A_n for each n such that

$$|f(t) - \varphi(t)| < A_n \mu^n, \ (\mu > 0, \ t \geq \gamma). \tag{37}$$

Thus, equation (35) asserts that the approximation becomes better the smaller μ is taken and holds exactly in the limit as $\mu \longrightarrow 0^+$. (Reference 2 gives a good exposition of this type of analysis.)

Assume that there is a switching curve in the $x\dot{x}$-phase plane of the form shown in the figure. If a switch is made at $(x_o, \dot{x}_o)$ from a $\ominus$-position to $\oplus$ for the forcing function, the quantity $\ddot{x}_o$ will in general be arbitrary and will cause a deviation from the switching curve. Indeed, the system is of third order and will not usually follow this projection in the $x\dot{x}$-plane. This deviation may drive the phase point back into the $\ominus$-region so that it is possible to have several switches on the way to the origin. For analysis, assume that $\ddot{x}_o$ is such that the trajectory remains in the $\oplus$-switching region and so an error $\epsilon(t)$ is defined to be

$$\epsilon(t) = x_o(\sigma) - x(t), \tag{38}$$

where the switching curve has been parameterized by σ so that $x_o(0)$ is the original initial value x_o and $\dot{x}_o(\sigma) = \dot{x}(t)$. Assuming that

$$\epsilon(t) > 0, \ 0 \leq t \leq t_o, \text{ and } \dot{x}(t_o) = 0,$$

we find

$$\begin{aligned} \Delta = \epsilon(t_o) - x_o(\sigma_o) \approx \tau k \log\left[\frac{\tau-\mu}{\tau} \cdot \frac{k}{k-\dot{x}_o-\mu\ddot{x}_o}\right] \\ - \left[x_o - (\tau+\mu)(k-\dot{x}_o) + \tau\mu\ddot{x}_o\right] - \tau k, \end{aligned} \tag{39}$$

or after manipulating with logarithms

$$\Delta \approx \mu k + \tau k \log\left(1-\frac{\mu}{\tau}\right) - \tau\mu\ddot{x}_o - \tau k \log\left(1-\frac{\mu\ddot{x}_o}{k-\dot{x}_o}\right)$$
$$- x_o - (\tau+\mu)\,\dot{x}_o - \tau k \log\left(1-\frac{\dot{x}_o}{k}\right). \tag{40}$$

Since nothing can be done with the terms in $\ddot{x}_o$,

$$x_o = -(\tau+\mu)\,\dot{x}_o - \tau k \log\left(1-\frac{\dot{x}_o}{k}\right) + x_o(\sigma_o) \tag{41}$$

is chosen as the switching curve. There will be a dead zone of half-length $x_o(\sigma_o)$ along the x-axis (usually $x_o(\sigma_o) = 0$) and the trajectory will miss the origin by an error of

$$\Delta \approx \mu k + \tau k \log\left(1-\frac{\mu}{\tau}\right) - \tau\mu\ddot{x}_o - \tau k \log\left(1-\frac{\mu\ddot{x}_o}{k-\dot{x}_o}\right) - x_o(\sigma_o). \tag{42}$$

Note that as $\mu \longrightarrow 0$ the exact formula for the second order case, namely

$$x_o = -\tau\left[\dot{x}_o + k \log\left(1-\frac{\dot{x}_o}{k}\right)\right],\ (\dot{x}_o < 0,\ x_o(\sigma_o) = 0), \tag{43}$$

is recovered from equation (41).

LOVINGOOD'S TECHNIQUE

The transformed equation is

$$x(\tau s + 1)(\mu s+1)X = ku/s. \tag{44}$$

The method of reference 1 is to introduce a new dependent variable $y(t)$,

$$Y = (\mu s+1)\ X \tag{45}$$

and to control it time-optimally. The equation for $y(t)$ is obviously

$$s(\tau s + 1)Y = ku/s \tag{46}$$

and the switching curve is

$$y_o = -\tau\left[\dot{y}_o + k \log\left(1-\frac{\dot{y}_o}{k}\right)\right],\ \dot{y}_o < 0. \tag{47}$$

Since $y = x + \mu\dot{x}$, this gives

$$x_o = -(\mu + \tau)\,\dot{x}_o - \tau\mu\,\ddot{x}_o - \tau k \log\left(1-\frac{\dot{x}_o+\mu\,\ddot{x}_o}{k}\right) \tag{48}$$

If the $\ddot{x}_o$-terms are simply ignored we return to the previous formula (41).

Thus, the method of asymptotic analysis of the equations of motion for this problem leads to the same control found by J. A. Lovingood in previous work.

CONCLUSIONS

An advantage the asymptotic approximation method appears to offer over other methods is its capability for estimating errors.

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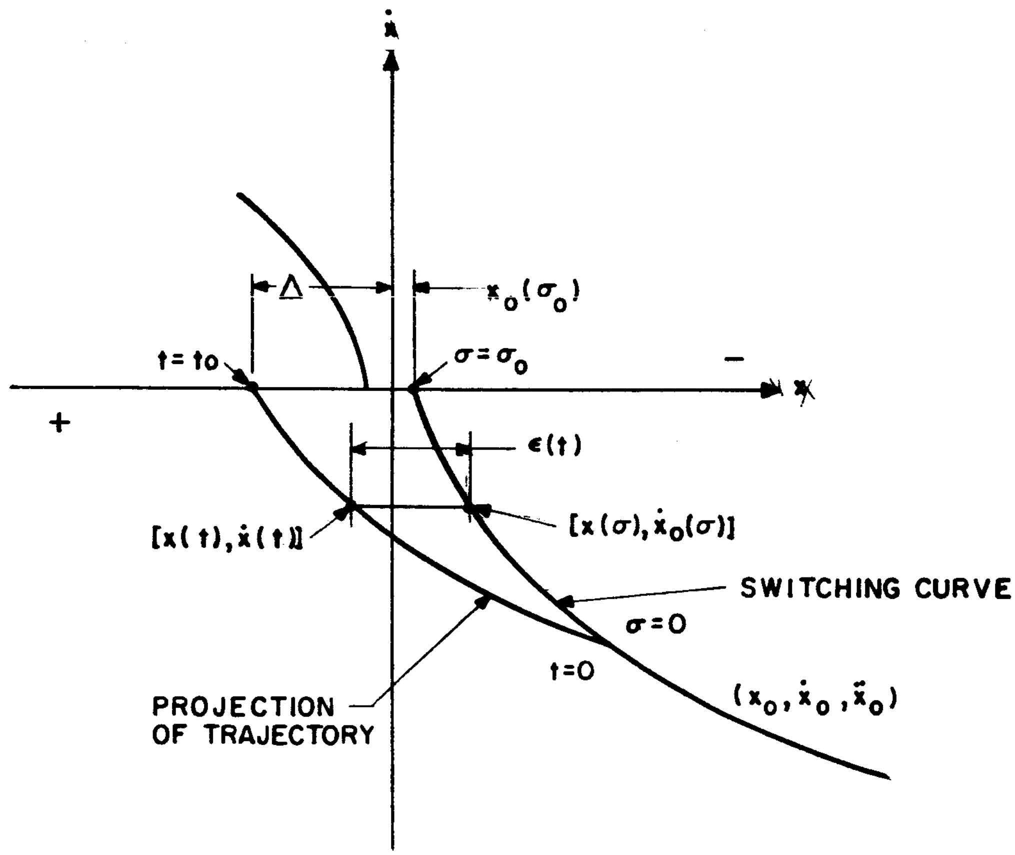

Figure 1. The Switching Curve